5. HALL SUBGROUPS

§5.1. Definition of a Hall Subgroup

A Sylow p-subgroup of a finite group G is a subgroup H where $|H| = p^n$ for some n and $|G:H|$ is coprime to p. The concept of a Hall subgroup generalizes this and some, but not all, of the Sylow theorems carry across to Hall subgroups.

Let Π be any set of primes and let $\boldsymbol{\Pi'}$ be its complement in the set of all primes. Define $\mathbf{N}_{\Pi}$ denote the set of all positive integers whose prime divisors are all in Π. So, $N_{\Pi'}$ is the set of all positive integers none of whose prime divisors are in Π. Clearly $N_{\Pi} \cap N_{\Pi'} = \{1\}$.

Example 1: If $\Pi = \{2, 5\}$ then $100 \in N_{\Pi}$; $21 \in N_{\Pi'}$ and 24 is contained in neither.

A **Hall Π-subgroup** of a finite group G is a subgroup H where $|H| \in N_{\Pi}$ and $|G:H| \in N_{\Pi'}$.

A Sylow p-subgroup is a Hall Π-subgroup where $\Pi = \{p\}$.

Example 2: Each of the 5 subgroups of $\mathbf{S}_5$ that are isomorphic to $\mathbf{S}_4$ are Hall $\{2, 3\}$-subgroup of S_5. A Hall $\{3, 5\}$-subgroup of $\mathbf{S}_5$ would have to have order 15. The only group of order 15 is $\mathbf{C}_{15}$ and clearly $\mathbf{S}_5$ has no elements of order 15.

This example shows that Hall Π-subgroups might not exist for some sets of primes. Technically we need to consider all sets of primes Π, but if Π contains any primes that don't divide the order of the group these irrelevant primes may be removed and the Hall Π-subgroups, if any exist, will be the same. Therefore we need only consider subsets of the set of primes dividing the order of the group and we now use Π′ to denote the complement of Π within that set of primes. Also, if Π or Π′ = ∅ the Hall subgroup will be the trivial subgroup or the whole group, so we only need to consider the proper non-trivial subsets.

So for $\mathbf{S}_5$ we consider only $\Pi_1 = \{2\}$, $\Pi_2 = \{3\}$, $\Pi_3 = \{5\}$, $\Pi_4 = \{2, 3\}$, $\Pi_5 = \{2, 5\}$ and $\Pi_6 = \{3, 5\}$. Hall Π-subgroups exist in $\mathbf{S}_5$ for Π_1 to Π_4 but not Π_5 and Π_6.

§5.2. Hall Subgroups of Finite Soluble Groups

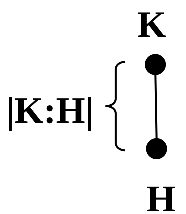
There is a deep theorem, that relies on representation theory, which states that every group of order $p^a q^b$, where p, q are primes, is soluble. For such a group Hall Π-subgroups exist for all sets of primes Π. For $\Pi = \{p\}$ and $\Pi = \{q\}$ they are the Sylow subgroups, for $\Pi = \varnothing$ it is the trivial subgroup and for $\Pi = \{p, q\}$ it is the whole group. (Other sets of primes, containing irrelevant primes, are equivalent to one of these four.)

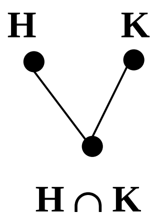
In fact having Hall subgroups for all sets of primes characterises soluble groups. We shall prove that a finite soluble group has at least one Hall subgroup for all Π and that any finite group having this property must be soluble. The theorem about groups of order $p^a q^b$ being soluble would then follow as a special case. However this doesn't provide an alternative proof because that theorem is used in the proof.

The proof in each direction involves a rather intricate proof by induction. In each case we assume that it holds for all smaller groups. This means that if we have a proper subgroup, or a quotient by a non-trivial normal subgroup, we can assume the result. Crucial to this process are the facts that subgroups and quotient groups of soluble groups are soluble and the fact that if both G/H and H are soluble then so is G.

The proofs lend themselves to a pictorial proof, where we draw at each stage a portion of the lattice of subgroups. If a subgroup, K, is lower than a subgroup H and is joined by a line, then H ≤ K. Alongside we show the index of H in K.

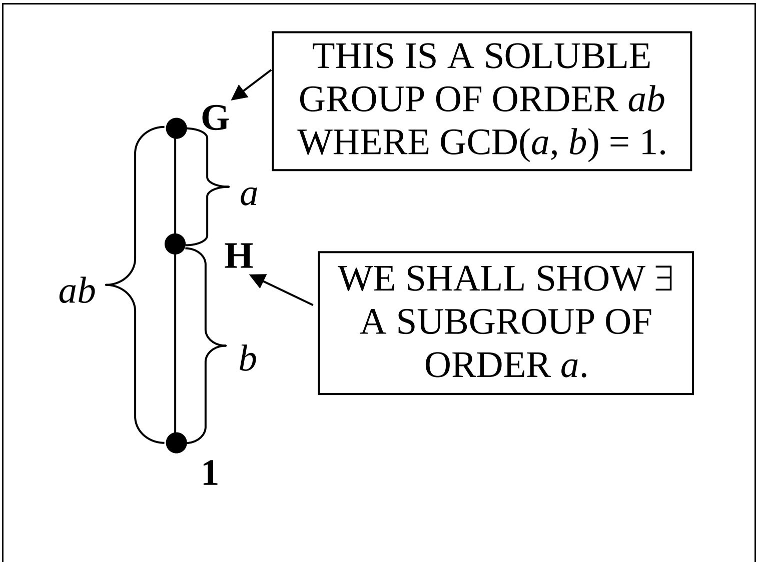
Also, if a subgroup is shown as being a common subgroup to H and K it is assumed to be H ∩ K.

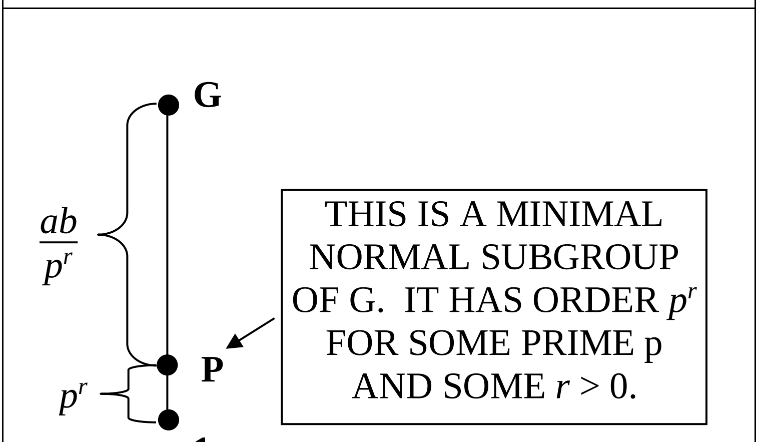




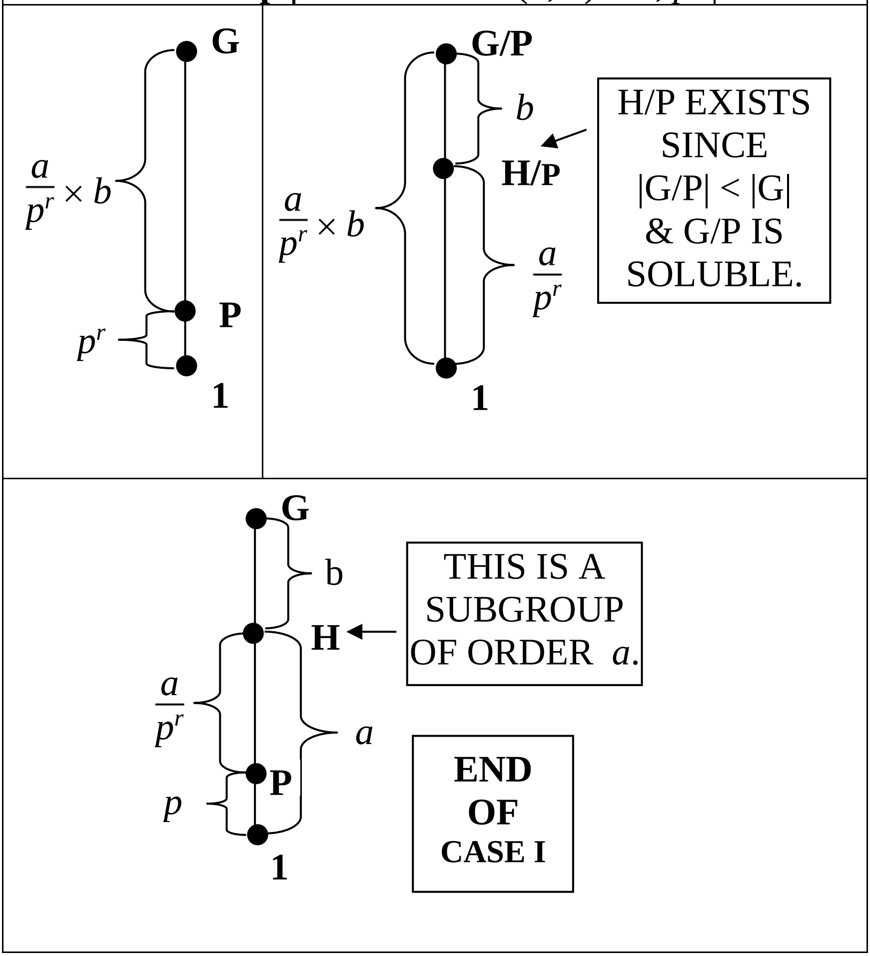
Theorem 1: A finite soluble group has at least one Hall Π-subgroup for every set of primes Π.

Proof: (Because of the graphic nature of this proof there will be a lot of blank space on the following pages.)

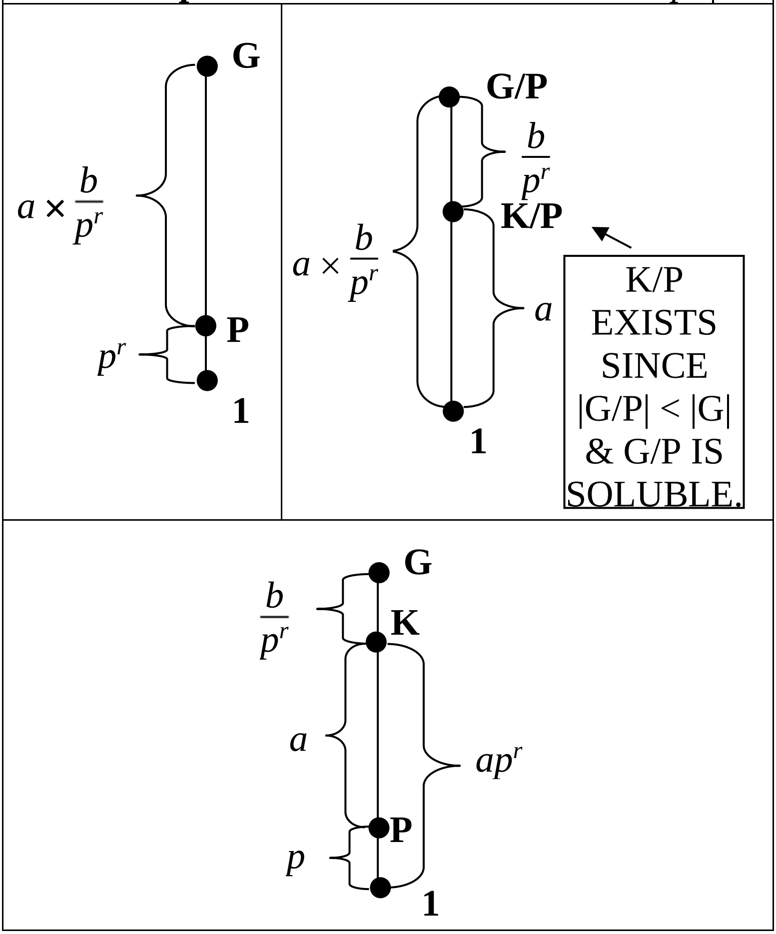




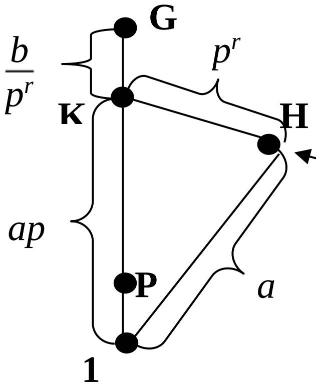
CASE I: p | a. Since GCD(a, b) = 1, $p^r \mid a$.



CASE II: *p* does not divide *a*. In this case $p^r \mid b$.



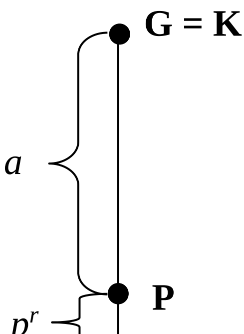
CASE IIA: K < G

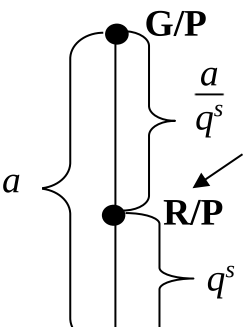


H EXISTS SINCE |K| < |G| & K IS SOLUBLE.

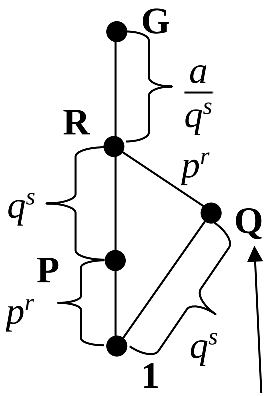
END OF CASE IIA.

CASE IIB: K = G





MINIMAL NORMAL SUBGROUP OF G/P



Q IS A SYLOW q-SUBGROUP OF R.

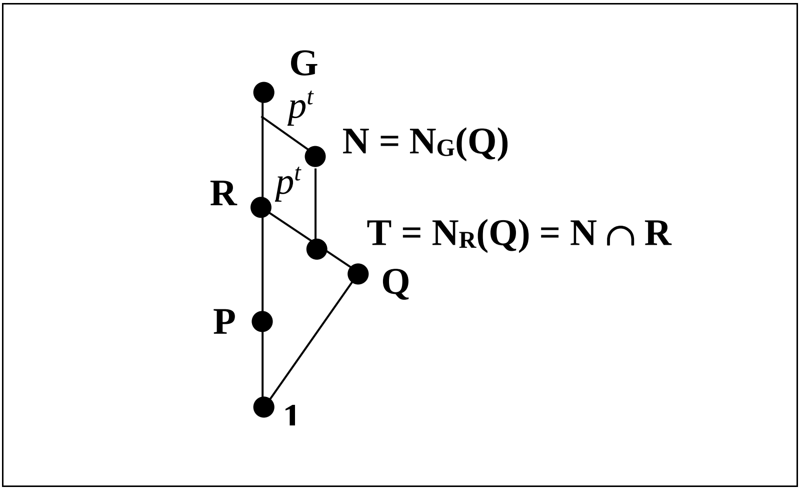
Every conjugate of Q in G is contained in R (since R is normal in G) and so is a Sylow q-subgroup of R and hence is conjugate to Q in R.

Thus #conjugates of Q in R = #conjugates of Q in G.

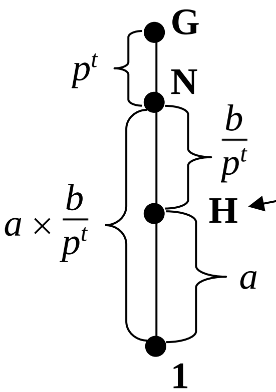
Hence

$|G:N_G(Q)| = |R:N_R(Q)|$

$= p^t$ for some t.


G
p^t
N = NG(Q)
R
p^t
T = NR(Q) = N ∩ R
Q
P
1

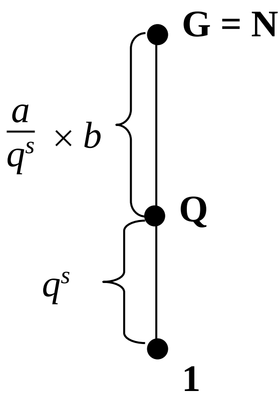
CASE IIB(i): N < G

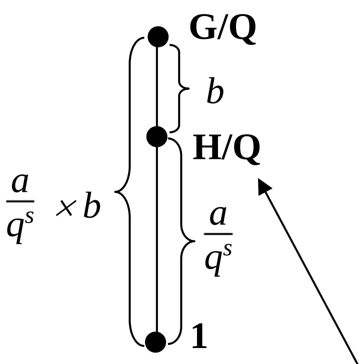


H EXISTS SINCE |N| < |G| AND N IS SOLUBLE.

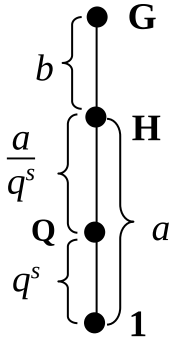
END OF CASE IIB(i)

CASE IIB(ii): N = G





H/Q EXISTS
SINCE |G/Q| < |G|
AND G/Q IS SOLUBLE.

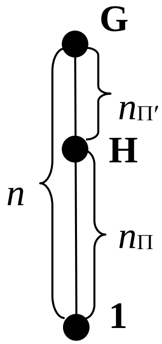


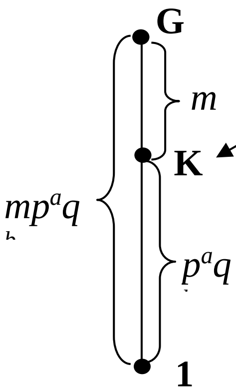
END OF CASE IIB(ii).
END OF PROOF.

✋☺

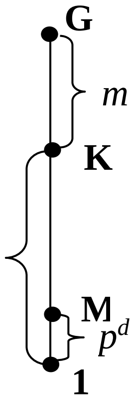
Theorem 2: If G has order $p^a q^b$, where p, q are primes, then G is soluble.
Proof: As explained above, we omit the proof. ☺

Theorem 3: If the finite group G has a Hall Π-subgroup for all Π then G is soluble.
Proof: (Again, because of the need not to split diagrams there will be a lot of blank space. Please turn over the page.)

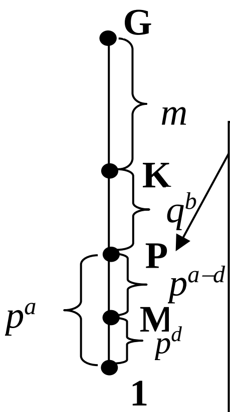
	WE SHALL SHOW THAT IF H EXISTS FOR ALL Π THEN G IS SOLUBLE.	WE PROVE IT BY INDUCTION SUPPOSE the theorem holds for smaller groups.
		IF |G| = p^a OR $p^a q^b$ then by Theorem 2, G is soluble.
		SUPPOSE |G| = $p^a q^b m$ where $m > 1$ and is coprime to both p and q.



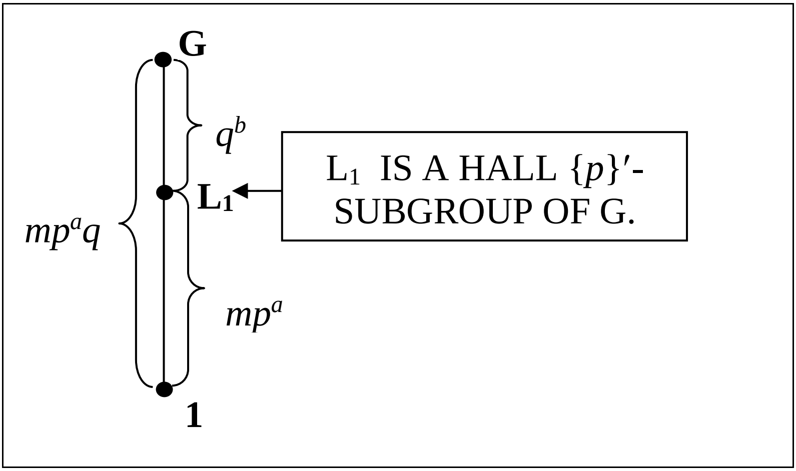
K IS A HALL {p, q}-SUBGROUP OF G.

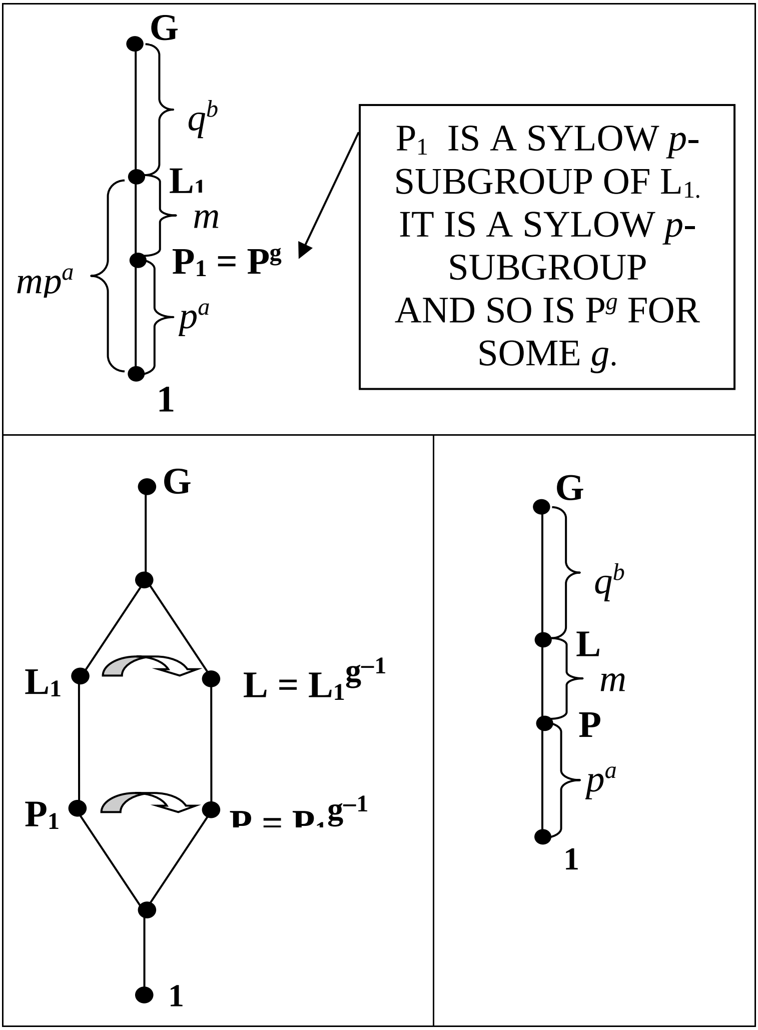


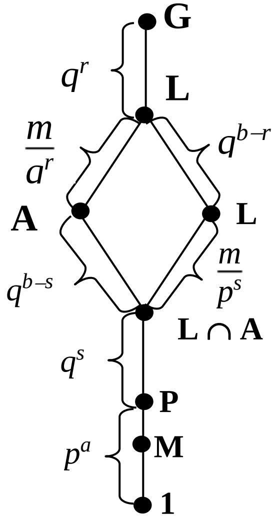
M IS A MINIMAL NORMAL SUBGROUP OF G. $|M| = p^d$ FOR SOME d.

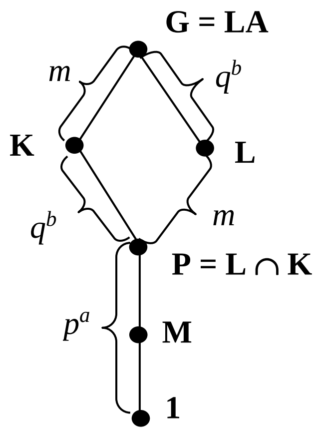


P IS A SYLOW p-SUBGROUP OF K CONTAINING M. IT IS A SYLOW p-SUBGROUP OF G SINCE GCD(p, m) = 1.

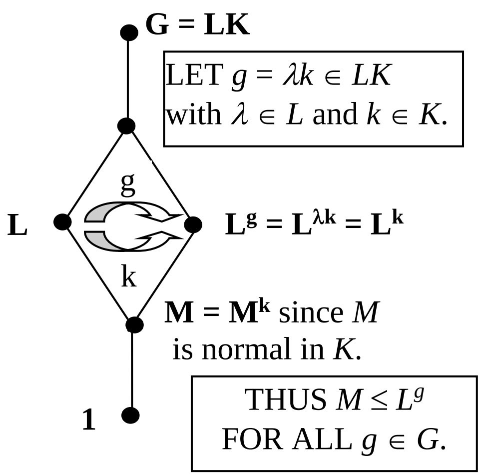

G
q^b
L_1
mp^aq
mp^a
1
L_1 IS A HALL $\{p\}'$-
SUBGROUP OF G.

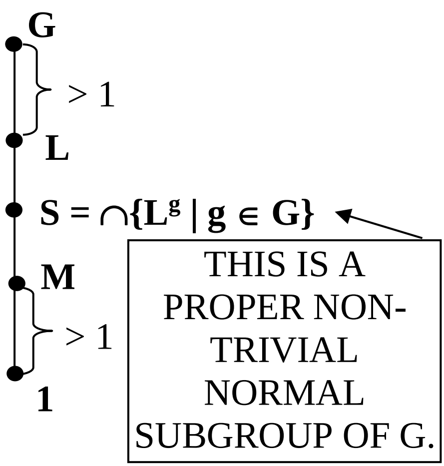

G
q^b
L₁
m
mp^a
P₁ = P^g
p^a
1
P₁ IS A SYLOW p-SUBGROUP OF L₁. IT IS A SYLOW p-SUBGROUP AND SO IS P^g FOR SOME g.
G
L₁
L = L₁^{g−1}
P₁
P = P₁^{g−1}
1
G
q^b
L
m
P
p^a
1

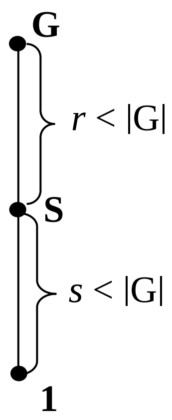




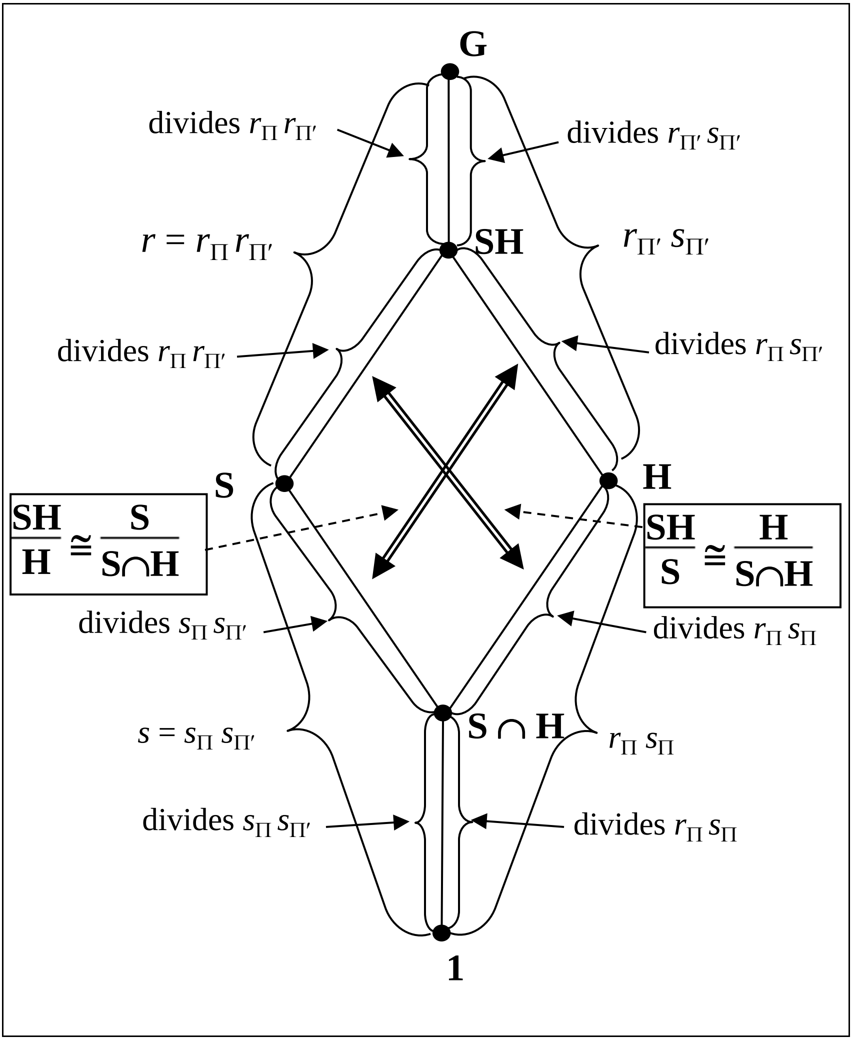
SINCE GCD(m, q) = 1:
$LA = G$ AND $L \cap A = P$.



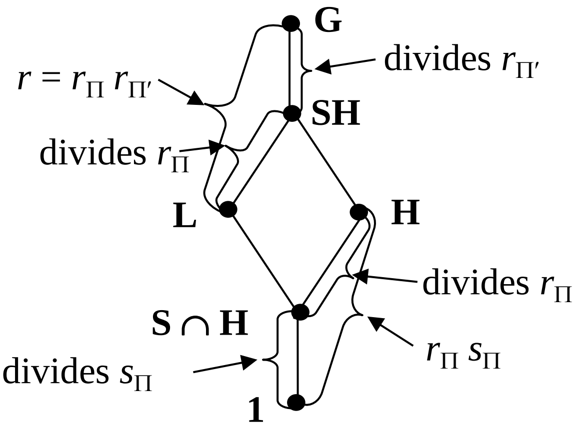


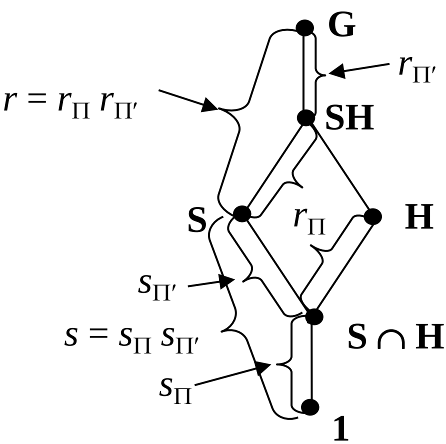


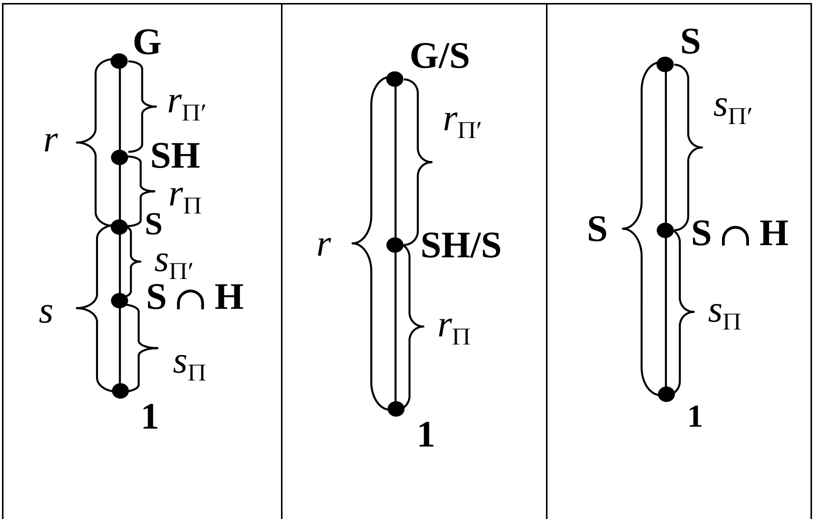
Let Π be any set of primes and let H be a Hall Π-subgroup of G.
Then $|H| = (rs)_{\Pi} = r_{\Pi}s_{\Pi}$ and
$|G:H| = (rs)_{\Pi'} = r_{\Pi'}s_{\Pi'}$.


G
divides $r_\Pi r_{\Pi'}$
divides $r_{\Pi'} s_{\Pi'}$
$r = r_\Pi r_{\Pi'}$
SH
$r_{\Pi'} s_{\Pi'}$
divides $r_\Pi r_{\Pi'}$
divides $r_\Pi s_{\Pi'}$
S
H
$\frac{SH}{H} \cong \frac{S}{S \cap H}$
$\frac{SH}{S} \cong \frac{H}{S \cap H}$
divides $s_\Pi s_{\Pi'}$
divides $r_\Pi s_\Pi$
$s = s_\Pi s_{\Pi'}$
S ∩ H
$r_\Pi s_\Pi$
divides $s_\Pi s_{\Pi'}$
divides $r_\Pi s_\Pi$
1

SINCE GCD(r_{Π}, $s_{\Pi'}$) = GCD($r_{\Pi'}$, s_{Π}) = 1 ……







THUS G/S AND *S* HAVE HALL Π-SUBGROUPS FOR ALL Π.	BY INDUCTION THEY ARE SOLUBLE.	HENCE G IS SOLUBLE. 🖐☺

§5.3. Supersoluble Groups

A **soluble** group is one such that, for some n, the n^{th} derived subgroup $G^{(n)}$ is trivial. Hence a group is soluble if and only if it has a finite normal series

$$G_0 = 1 < G_1 < G_2 < \ldots < G_n = G$$

where each G_i is normal in G and where each G_{i+1}/G_i is abelian.

A group is **supersoluble** if and only if it has such a finite *normal* series where each G_{i+1}/G_i is cyclic.

Example 3: $\mathbf{S}_3$ is supersoluble, but $\mathbf{S}_4$ is not (though it is soluble).
The normal series for $\mathbf{S}_3$ (it has only one) is $1 < \mathbf{A}_3 < \mathbf{S}_3$, where $\mathbf{A}_3/1 \cong \mathbf{C}_3$ and $\mathbf{S}_3/\mathbf{A}_3 \cong \mathbf{C}_2$.

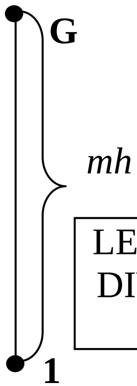
The only normal series for $\mathbf{S}_4$ is $1 < \mathbf{V}_4 < \mathbf{A}_4 < \mathbf{S}_4$ where
$$\mathbf{V}_4 = \{I, (12)(34), (13)(24), (14)(23)\}.$$
$\mathbf{S}_4/\mathbf{A}_4 \cong \mathbf{C}_2$ and $\mathbf{A}_4/\mathbf{V}_4 \cong \mathbf{C}_3$ but $\mathbf{V}_4/1 \cong \mathbf{C}_2 \times \mathbf{C}_2$ and so is not cyclic. There is no normal subgroup that can be placed between 1 and $\mathbf{V}_4$ to break the $\mathbf{C}_2 \times \mathbf{C}_2$ into two $\mathbf{C}_2$'s.

Lagrange's theorem states that the order of any subgroup of a finite group divides the order of the group. The converse does not hold in general. For example, $\mathbf{A}_4$ has no subgroup of order 6.

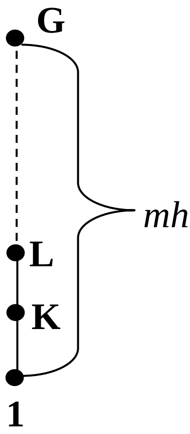
A finite group, G, that has at least one subgroup order m for each m dividing $|G|$ is called a **CLT group** (with 'CLT' standing for 'converse to Lagrange's Theorem').

Theorem 4: Finite supersoluble groups are CLT groups.
Proof: (next page)

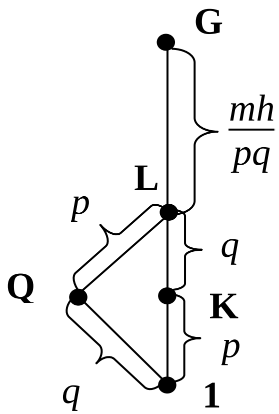
INDUCTION
SUPPOSE TRUE FOR SMALLER GROUPS



LET *m* BE A DIVISOR OF |G|.

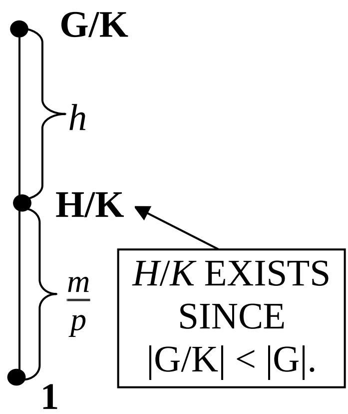


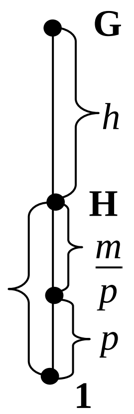
LET
1 < K < L < …BE A
CHIEF SERIES FOR G.



THEN |K|, |L/K| ARE PRIME

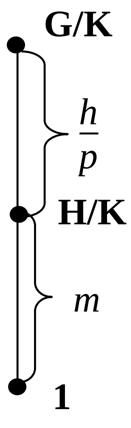
CASE 1: *p* | *m*.

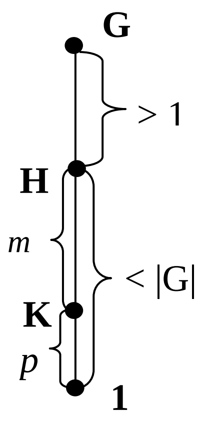


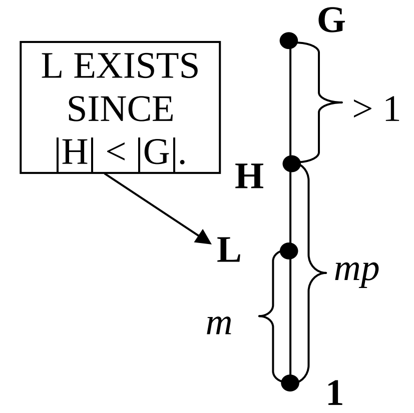


CASE 2: *p* does not divide *m* and *mp* < *n*.

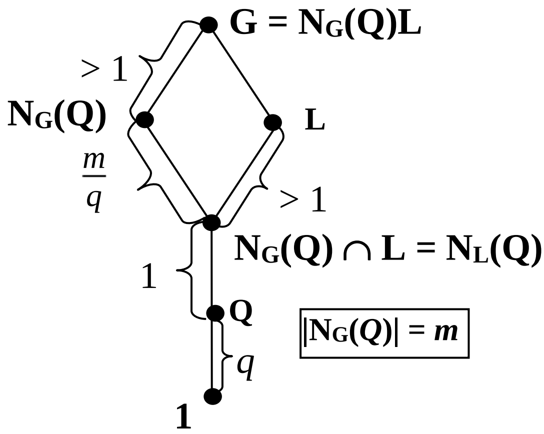
Hence $p|h$.



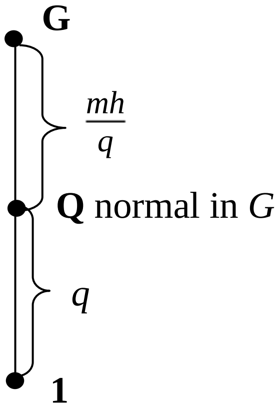


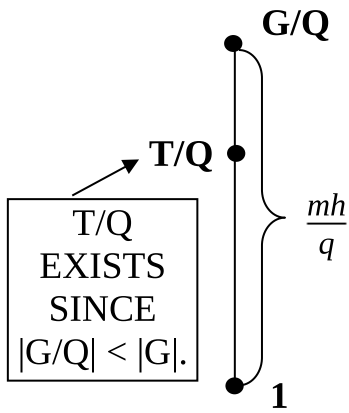


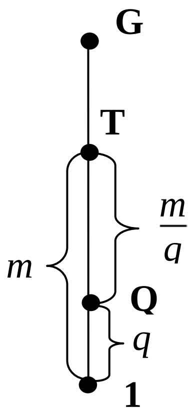
CASE 3: *p* does not divide *m*, *mp* = *n* and $N_G(Q) < G$.



CASE 4: p does not divide *m*, *mp* = *n* and $N_G(Q) = G$.







EXERCISES FOR CHAPTER 5

Exercise 1: For each of the following statements determine whether it is true or false.

(1) S_3 is a Hall subgroup of S_4.

(2) If $\Pi = \{3, 7\}$ then $1323 \in N_\Pi$.

(3) Finite soluble group have Hall Π-subgroups for all Π.

(4) If $|G| = 1323$ then G is soluble.

(5) Finite dihedral groups are supersoluble.

(6) If G is a finite supersoluble group of order $1323k$ then G has a subgroup of order 1323.

Exercise 2: Prove that if a finite group G has a soluble Hall Π-subgroup then it has a Hall subgroup for every subset of Π.

Exercise 3: For each of the following sets of prime Π determine whether of not $\mathbf{S}_6$ has a Hall Π-subgroup:

(a) $\Pi = \{5\}$;

(b) $\Pi = \{3, 5\}$;

(c) $\Pi = \{2, 3\}$ [**HINT:** Suppose S_6 has a subgroup H of index 5 and consider the permutation on the right cosets of H by right multiplication. Use the fact that the only normal subgroups of $\mathbf{S}_6$ are **1**, $\mathbf{A}_6$ and $\mathbf{S}_5$.].

SOLUTIONS FOR CHAPTER 5

Exercise 1:
(1) FALSE $|\mathbf{S}_3|$ and $|\mathbf{S}_4/\mathbf{S}_3|$ are both even.
(2) TRUE: $1323 = 3^3.7^2$.
(3) TRUE
(4) TRUE: All groups of order $p^a q^b$ are soluble.
(5) TRUE: $D_{2n} = \langle A, B \mid A^n, B^2, BA = A^{-1}B\rangle$.
Every subgroup of $\langle A\rangle$ is normal in D_{2n} so a normal series for $\langle A\rangle$ with cyclic quotients, followed by D_{2n} itself, would be a normal series for D_{2n} with cyclic quotients.
(6) TRUE: Finite supersoluble groups have subgroups of all orders dividing their order.

Exercise 2: Let G have a Hall Π-subgroup H and suppose H is soluble. Let $\Theta \subseteq \Pi$. Then H has a Hall Θ-subgroup K with $|K| \in N_\Theta$. Then $|H:K| \in N_{\Pi-\Theta'} \subseteq N_{\Theta'}$.
Now $|G:H| \in N_{\Pi'} \subseteq N_{\Theta'}$, so $|G:K| = |G:H|.|H:K| \in N_{\Theta'}$.

Exercise 3:
(a) A Hall Π-subgroup is simply a Sylow 5-subgroup, which certainly exists for all finite groups.
(b) Let H be a Hall {3, 5}-subgroup of $\mathbf{S}_6$.
Then $|H| = 45 = 3^2.5$.
Now H has at least one Sylow 3-subroups and at least one Sylow 5-subgroup. The number of Sylow 5-subgroups must have the form $1 + 5k$ and divides 9, so it must be 1.

So there is just one Sylow 5-subgroup and it must contain all the elements of order 5. Hence $\mathbf{S}_6$ has only 4 elements of order 5, which is clearly a contradiction. Hence $\mathbf{S}_6$ has no Hall {3, 5}-subgroup.

(c) Suppose that H is a Hall {2, 3}-subgroup of $\mathbf{S}_6$.

Then |H| = 144 and $|S_6:H| = 5$.

If a is any 5-cycle then the cosets must be:

$$H, Ha, Ha^2, Ha^3, Ha^4.$$

Let G act on the right cosets of H by $Hx \to Hxg$.

In other words let $\Omega:\mathbf{S}_6 \to \mathbf{S}_5$ be defined by

$$\Omega(g) = \{Hx \to Hxg\}.$$

Clearly $\Omega(a)$ must be a 5-cycle. Hence $|\text{im } \Omega| \geq 5$.

Let $K = \ker \Omega$.

By the 1st Isomorphism Theorem, $|G/K| \geq 5$ and so $|K| \leq 144$.

Since K is normal in $\mathbf{S}_6$ it K = **1**.

Hence $\mathbf{S}_6$ is isomorphic to a subgroup of $\mathbf{S}_5$, a contradiction. So no such Hall {2, 3}-subgroup exists.